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A New Model to Describe Quarkonium Systems under Modified Cornell Potential at Finite Temperature in pNRQCD

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Abstract. In the present work, the three-dimensional modified radial Schrödinger equation is analytically solved. The nonrelativistic interactions under new modified Cornell potential (NMCP, in short) at finite temperature, are extended to the symmetries of nonrelativistic noncommutative space phase (NRNSP, in short), using the generalized Bopp's shift method in the case of perturbed nonrelativistic quantum chromodynamics (pNRQCD). W generalize this process by adding multi-variable coupling potentials $\frac{b}{2r^3}\vec{L}\vec{\Theta}$, $\frac{C}{2r}\vec{L}\vec{\Theta}$ and $\left(-D\vec{L}\vec{\Theta}+\frac{\mathbf{L}\bar{\boldsymbol{\theta}}}{2\mu}\right)$ together with the modified Cornell potential model in three-dimensional nonrelativistic quantum mechanics noncommutative phase space (3D-NCSP, in short). The new energy eigenvalues and the corresponding Hamiltonian operator are calculated in 3D-NCSP symmetries instead of solving the modified Schrödinger equation with the Weyl Moyal star product. The present results, in (3D-NCSP), are applied to the charmonium and bottomonium masses at finite temperature. The present approach successfully generalizes the energy eigenvalues at finite temperature in 3D-NCSP symmetries. It is found that the perturbative solutions of the discrete spectrum and quarkonium mass can be expressed by the Gamma function, the discreet atomic quantum numbers (j,l,s,m) of the $Q\overline{Q}$ state and the potential parameters (A,b,C,D), in addition to noncommutativity parameters (Θ and $\bar{\theta}$). The total complete degeneracy of new energy levels of NMCP changed to become equals to the value $6n^2$ instead the values $2n^2$ in ordinary quantum mechanics. Our obtained results are in good agreement with the already existing literature in NCSP.

Subject Classification Numbers: 03.65.-w; 03.65. Ge; 03.65. Fd; 03.65. Ca

1. Introduction

It is well known that the Cornell potential model was one of the most popular models for studying the interactions in such systems as quarkonium (heavy quarkonia) consisting of heavy quark and antiquark (charmonium$c\bar{c}$, bottomonium $b\bar{b}$ and $b\bar{c}$ mesons in the nonrelativistic quantum chromodynamics NRQCD). It is consisting of two terms. One of the terms is responsible for the Coulomb interaction of quarks and the other corresponds to the strong interaction, which provides confinement, it is the first potentials proposed to describe the interaction between heavy quarks [1-10]. Furthermore, the Cornell model has been extraordinarily successful in describing hadronic phenomenology, in addition to its success in describing a huge amount of experimental data including masses, widths, radiative and strong transitions [11-12]. The problem of calculating the energy spectra for the Schrodinger equation with various types of potentials such as Cornell potential and the Killingbeck potential at finite temperature has been attracting interest for recent years [9-11]. The main objective is to develop the research article for Abu-Shady [11] and expand it to a large symmetry known by the noncommutative space phase (NCSP, in short) to achieve a more accurate physical vision so that this study becomes valid in the field of nanotechnology. And on the other hand, to explore the possibility of creating new applications and more profound interpretations in the sub-atomics and nano scales using the new version of the modified Cornell potential, which has the following form:

$$V_{cp}(r) = \underbrace{a(T,r)r - \frac{b(T,r)}{r}}_{\text{Ordinary}-\text{QM}} \rightarrow V_{cp}(\hat{r}) = \underbrace{V_{cp}(r) - \left(\frac{C}{r^4} + \frac{G}{2r^3} + \frac{F}{2r}\right)\vec{L}\vec{\Theta}}_{NCQM} \tag{1}$$

We refer to this term $\vec{\boldsymbol{L}}\vec{\boldsymbol{\Theta}}$ in the materials and methods section. The new structure of NCSP based on new canonical commutations relations in both Schrödinger SP and Heisenberg HP, respectively, as follows (Throughout this paper, the natural units $\boldsymbol{c} = \boldsymbol{\hbar} = \mathbf{1}$ will be used) [13-18]:

$$[x_\mu, p_\nu] = i\delta_{\mu\nu} \rightarrow [\hat{x}_\mu \overset{*}{,} \hat{p}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)] = i\delta_{\mu\nu}\hbar_{eff} \Rightarrow \Delta\hat{x}_\mu \Delta\hat{p}_\nu \geq \frac{\delta_{\mu\nu}}{2}$$
$$[x_\mu, x_\nu] = 0 \rightarrow [\hat{x}_\mu \overset{*}{,} \hat{x}_\nu] = [\hat{x}_\mu(t) \overset{*}{,} \hat{x}_\nu(t)] = i\theta_{\mu\nu} \Rightarrow \Delta\hat{x}_\mu \Delta\hat{x}_\nu \geq \frac{\theta_{\mu\nu}}{2}$$
$$[p_\mu, p_\nu] = 0 \rightarrow [\hat{p}_\mu \overset{*}{,} \hat{p}_\nu] = [\hat{p}_\mu(t) \overset{*}{,} \hat{p}_\nu(t)] = i\overline{\theta}_{\mu\nu} \Rightarrow \Delta\hat{p}_\mu \Delta\hat{p}_\nu \geq \frac{\overline{\theta}_{\mu\nu}}{2} \tag{2}$$

where the indices $\mu, \nu \equiv \overline{1,3}$ and $\hbar_{eff}$ is the effective Planck constant. This means that the principle of uncertainty for Heisenberg generalized to include another two new uncertainties related to the positions $(\hat{x}_\mu, \hat{x}_\nu)$ and the momenta's $(\hat{p}_\mu, \hat{p}_\nu)$, in addition to the ordinary uncertainty of $(\hat{x}_\mu, \hat{p}_\nu)$. The very small two parameters $\theta^{\mu\nu}$ and $\overline{\theta}^{\mu\nu}$ (compared to the energy) are elements of two antisymmetric real matrixes, parameters of noncommutativity and $(*)$ denote to the Weyl Moyal star product, which is generalized between two arbitrary functions $(f, g)\,(x, p)$ to the new form $\hat{f}(\hat{x}, \hat{p})\hat{g}(\hat{x}, \hat{p}) \equiv (f * g)(x, p)$ in 3D-NCSP symmetries [18-25]:

$$(f, g)\,(x, p) \rightarrow (f * g)(x, p) = \left(fg - \frac{i}{2}\theta^{\mu\nu}\partial_\mu^x f \partial_\nu^x g - \frac{i}{2}\overline{\theta}^{\mu\nu}\partial_\mu^p f \partial_\nu^p g\right)(x, p) \tag{3}$$

The second and the third terms in the above equation are present the effects of (space-space) and (phase-phase) noncommutativity properties. It should be noted that noncommutativity was introduced firstly by Heisenberg in 1930 [26] and then by Syndre in 1947 [27]. However, the new operators $\hat{\xi}(t) = (\hat{x}_\mu \vee \hat{p}_\mu)(t)$ in HP are depending on the corresponding new operators $\hat{\xi} = \hat{x}_\mu \vee \hat{p}_\nu$ in SP from the following projections relations:

$$\xi(t) = exp(i\hat{H}_{cp}(t - t_0))\xi\, exp(-i\hat{H}_{cp}(t - t_0)) \Rightarrow$$
$$\hat{\xi}(t) = exp(i\hat{H}_{nc}^{cp}(t - t_0)) * \hat{\xi} * exp(-i\hat{H}_{nc}^{cp}(t - t_0)) \tag{4}$$

here $\xi = x_\mu \vee p_\nu$ and $\xi(t) = (x_\mu \vee p_\nu)(t)$, while the dynamics of new systems $\frac{d\xi(t)}{\mathrm{dt}}$ are described from the following motion equations in NCSP:

$$\frac{d\xi(t)}{\mathrm{dt}} = -i[\xi(t), \hat{H}_{cp}] \Rightarrow \frac{d\hat{\xi}(t)}{\mathrm{dt}} = -i[\hat{\xi}(t) \overset{*}{,} \hat{H}_{nc}^{cp}] \tag{5}$$

The operators $\hat{H}_{cp}$ and $\hat{H}_{nc}^{cp}$ are present the quantum Hamiltonian operators for MCP and NMCP in the QM and its extension NCSP, respectively. This paper consists of five sections and the organization scheme is given as follows: In the next section, the theory part, we briefly review the SE with modified Cornell potential at finite temperature on based to ref. [11-12]. Section 3 is devoted to studying the modified Schrödinger equation MSE by applying the generalized Bopp's shift method and obtaining the NMCP and the modified spin-orbit operator at finite temperature. Then, we applied the standard perturbation theory to find the quantum spectrum of the ground state, the first excited state and the n^{th}excited-state produced by the effects of modified spin-orbit and modified Zeeman interactions. After that, in the fourth section, a discussion of the main results is presented in addition to determining the new formula of mass spectra of the quarkonium system in 3D-NCSP symmetries. Finally, in the last section, a summary and conclusions are presented.

2. Theory

2.1 Overview of the eigenfunctions and the energy eigenvalues for the modified Cornell potential at finite temperature in QM

We shall recall briefly in this section, the time-independent Schrödinger equation for the MCP at finite temperature [11]:

$$V_{cp}(r) = ar - \frac{b}{r} \rightarrow V_{cp}(r) = a(T,r)r - \frac{b(T,r)}{r} \tag{6}$$

The relative spatial coordinate between the two quarks is$r$, a and b are purely phenomenological constants of the model. In a thermal medium of a positive temperature T, the potential is modified by color screening which can be parameterized in the form [7, 11]:

$$a(T,r) = \frac{A}{m_D(T)r}(1 - exp(-m_D(T)r)) \text{ and } b(T,r) = b\, exp(-m_D(T)r) \tag{7}$$

Here $m_D(T)$ is the Debye screening mass. The expand with Taylor series around $r = 0$ gives [11]:

$$V_{cp}(r) = A + Cr - \frac{b}{r} - Dr^2 \tag{8}$$

where $A = bm_D(T)$, C = a-$1/2bm_D{}^2(T)$ and D = $1/2am_D(T)$. If we insert this potential into the Schrödinger equation, the radial part function $R(r) = \frac{U(r)}{r}$ is given as [11, 12]:

$$\frac{d^2U(r)}{dr^2} + \frac{2}{r}\frac{dU(r)}{dr} + 2\mu\left[E - A + \frac{b}{r} - Cr + Dr^2 - \frac{l(l+1)}{2\mu r^2}\right]U(r) = 0$$
$$\rightarrow \frac{d^2R(r)}{dr^2} + 2\mu\left[E - A + \frac{b}{r} - Cr + Dr^2 - \frac{l(l+1)}{2\mu r^2}\right]R(r) = 0 \tag{9}$$

here $\mu = \frac{m_q m_{\bar{q}}}{m_q + m_{\bar{q}}}$ the reduced mass for the quarkonium particle for example $(c\bar{c}, b\bar{b}, c\bar{s}, b\bar{s}, b\bar{u})$ and $c\bar{b}$. The complete wave function $\Psi(r,\theta,\phi) = \frac{R(r)}{r}Y_l^m(\theta,\phi)$ is given by [11, 12]:

$$\Psi(r,\theta,\phi) = C_{nl} r^{-\frac{D_2}{\sqrt{2D_{1n}}}-1} exp\left(\sqrt{2D_{1n}}r\right)\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n+\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(-2\sqrt{2D_{1n}}r\right)\right)Y_l^m(\theta,\phi) \tag{10}$$

In addition, the energy E_{nl} of the potential in Eq. (8) [11]:

$$E_{nl} = A + \frac{3C}{\delta} - \frac{6D}{\delta^3} - \frac{2\mu\left(\frac{3C}{\delta^2} - \frac{8D}{\delta^3} + b\right)}{\left[1 + 2n \pm \sqrt{1 + 4((l+1/2)^2 - 1/4) + \frac{8\mu C}{\delta^3} - 24\frac{\mu D}{\delta^4}}\right]^2} \tag{11}$$

where $D_{1n} = -\mu\left(E_n - A - \frac{3C}{\delta} + \frac{6D}{\delta^2}\right)$, $D_2 = \mu\left(\frac{3C}{\delta^2} - \frac{8D}{\delta^3} + b\right)$ and C_{nl} is a normalization constant while $r_0 \equiv 1/\delta$ is the characteristic radius of meson and n is a natural number accounting for the radial excitation while l is a non-negative integer number that represents the orbital angular momentum.

3. Materials and Methods

3.1 Solution of MSE for new modified Cornell potential at finite temperature in pNRQCD

In this section, we shall give an overview or a brief preliminary for NMCP in (3D-NCSP) symmetries. To perform this task the physical form of MSE, it is necessary to replace ordinary three-dimensional Hamiltonian operators $\hat{H}_{cp}(x_\mu, p_\mu)$, complex wave function $\Psi(\vec{r})$ and energy E_{nl} by new three Hamiltonian operators $\hat{H}_{nc}^{cp}(\hat{x}_\mu, \hat{p}_\mu)$, the new complex wave function $\hat{\Psi}\left(\overleftrightarrow{\hat{r}}\right)$ and new values E_{nc}^{cp} respectively. In addition to replacing the ordinary product with the Weyl Moyal star product, which allows us to construct the MSE in 3D-NCSP symmetries as [28-31]:

$$\hat{H}_{cp}(x_\mu, p_\mu)\Psi(\vec{r}) = E_{nl}\Psi(\vec{r}) \Rightarrow \hat{H}_{nc}^{cp}(\hat{x}_\mu, \hat{p}_\mu) * \Psi\left(\vec{\hat{r}}\right) = E_{nc}^{cp}\Psi\left(\vec{\hat{r}}\right) \tag{12}$$

The generalized Bopp's shift method has been successfully applied to relativistic and nonrelativistic noncommutative quantum mechanical problems using modified Dirac equation (MDE), modified Klein-Gordon equation (MKGE) and MSE. This method has produced very promising results for several situations having physical, chemical interests. The method reduces MDE, MKGE and MSE to the Dirac equation, Klein-Gordon and equation Schrödinger, respectively, under two-simultaneously translations in space and phase. It is based on the following new commutators [13-17, 32,33]:

$$[\hat{x}_\mu, \hat{x}_\nu] = [\hat{x}_\mu(t), \hat{x}_\nu(t)] = i\theta_{\mu\nu} \text{ and } [\hat{p}_\mu, \hat{p}_\nu] = [\hat{p}_\mu(t), \hat{p}_\nu(t)] = i\overline{\theta}_{\mu\nu} \tag{13}$$

The new generalized positions and momentum coordinates $(\hat{x}_\mu, \hat{p}_\nu)$ in 3D-NCSP are defined in terms of the commutative counterparts (x_μ, p_ν) in ordinary quantum mechanics via, respectively [29-32]:

$$(x_\mu, p_\nu) \Rightarrow (\hat{x}_\mu, \hat{p}_\nu) = \left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu, p_\mu + \frac{\overline{\theta}_{\mu\nu}}{2}x_\nu\right) \tag{14}$$

The above equation allows us to obtain the two operators $(\hat{r}^2, \hat{p}^2)$ in 3D-NCSP symmetries [28-31]:

$$(r^2, p^2) \Rightarrow (\hat{r}^2, \hat{p}^2) = \left(r^2 - \vec{L}\vec{\Theta}, p^2 + \vec{L}\,\vec{\overline{\theta}}\right) \tag{15}$$

The two couplings $\vec{L}\vec{\Theta}$ and $\vec{L}\,\vec{\overline{\theta}}$ are $(L_x\Theta_{12} + L_y\Theta_{23} + L_z\Theta_{13})$ and $(L_x\overline{\theta}_{12} + L_y\overline{\theta}_{23} + L_z\overline{\theta}_{13})$, respectively and $(L_x$, L_y and $L_z)$ are the three components of the angular momentum operator $\vec{L}$ while $\Theta_{\mu\nu} = \theta_{\mu\nu}/2$. Thus, the reduced Schrödinger equation (without star product) can be written as:

$$\hat{H}(\hat{x}_\mu, \hat{p}_\mu) * \Psi\left(\vec{\hat{r}}\right) = E_{nc}^{cp}\Psi\left(\vec{\hat{r}}\right) \Rightarrow H(\hat{x}_\mu, \hat{p}_\mu)\psi(\vec{r}) = E_{nc}^{cp}\psi(\vec{r}) \tag{16}$$

The new operator of Hamiltonian $H_{nc}^{cp}(\hat{x}_\mu, \hat{p}_\nu)$ can be expressed as:

$$\begin{aligned} H_{cp}(x_\mu, x_\mu) \Rightarrow H_{nc}^{cp}(\hat{x}_\mu, \hat{p}_\mu) &\equiv H\left(\hat{x}_\mu = x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu, \hat{p}_\mu = p_\mu + \frac{\bar{\theta}_{\mu\nu}}{2}x_\nu\right) \\ &= \frac{\hat{p}^2}{2\mu} + V_{cp}\left(\hat{r} = \sqrt{\left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu\right)\left(x_\mu - \frac{\theta_{\mu\nu}}{2}p_\nu\right)}\right) \end{aligned} \tag{17}$$

Where $V_{cp}(\hat{r})$ denote to the NMCP in 3D-NCSP symmetries:

$$V_{cp}(r) \Rightarrow V_{cp}(\hat{r}) = A + C\hat{r} - \frac{b}{\hat{r}} - D\hat{r}^2 \tag{18}$$

Again, applying Eq. (15) to find the three terms ($C\hat{r}$, $\left(-\frac{b}{\hat{r}}\right)$ and $(-D\hat{r}^2)$), which will be used to determine the NMCP $V_{cp}(\hat{r})$, as follows:

$$\begin{cases} -\frac{b}{r} \to -\frac{b}{\hat{r}} = -\frac{b}{r} - \frac{b}{2r^3}\vec{L}\vec{\Theta} + O(\Theta^2) \\ Cr \to C\hat{r} = Cr - \frac{C}{2r}\vec{L}\vec{\Theta} + O(\Theta^2) \\ -Dr^2 \to -D\hat{r}^2 = -Dr^2 + D\vec{L}\vec{\Theta} + O(\Theta^2) \end{cases} \tag{19}$$

Substituting, Eq. (19) into Eq. (18), gives the new modified Cornell potential in 3D-NCSP symmetries as follows:

$$V_{cp}(\hat{r}) = V_{cp}(r) - \left(\frac{b}{2r^3} + \frac{C}{2r} - D\right)\vec{L}\vec{\Theta} \tag{20}$$

By making the substitution above equation into Eq. (17), we find the global our working new modified Hamiltonian operator $H_{nc}^{cp}(\hat{r})$ satisfies the equation in 3D-NCSP symmetries:

$$H_{cp}(x_\mu, p_\nu) \Rightarrow \widehat{H}_{nc}^{cp}(\hat{r}) = H_{cp}(x_\mu, p_\nu) + H_{pert}^{cp}(r, \Theta, \bar{\theta}) \tag{21}$$

where the operator $H_{cp}(x_\mu, p_\nu)$ is just the ordinary Hamiltonian operator with modified Cornell potential in commutative quantum mechanics:

$$H_{cp}(x_\mu, p_\mu) = \frac{p^2}{2\mu} + A + Cr - \frac{b}{r} - Dr^2 \tag{22}$$

while the rest part $H_{pert}^{cp}(r, \Theta, \bar{\theta})$ (The perturbative Hamiltonian operator) is proportional with two infinitesimals parameters (Θ and $\bar{\theta}$):

$$H_{pert}^{cp}(r, \Theta, \bar{\theta}) = -\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right)\vec{L}\vec{\Theta} + \frac{\vec{L}\vec{\bar{\theta}}}{2\mu} \tag{23}$$

Thus, we can consider $\boldsymbol{H}_{\text{per-}\boldsymbol{cp}}(\boldsymbol{r})$ as a perturbation term compared with the principal Hamiltonian operator $\boldsymbol{H}_{\boldsymbol{cp}}(\boldsymbol{x}_\mu, \boldsymbol{p}_\mu)$ in 3D-NCSP symmetries.

3.2 The exact modified spin-orbit operator for heavy quarkonium systems under NMCP in pNRQCD:

In this subsection, we will apply the same strategy, which we have seen exclusively in some of our published scientific works [29-31]. Under such a particular choice, one can easily reproduce both couplings ($\vec{L}\vec{\Theta}$ and $\vec{L}\vec{\bar{\theta}}$) to the new physical forms ($\gamma\Theta\vec{L}\vec{S}$ and $\gamma\bar{\theta}\vec{L}\vec{S}$), respectively. Thus, the perturbative Hamiltonian operator $H_{pert}^{cp}(r, \Theta, \bar{\theta})$ for heavy quarkonium system will be transformed into the modified spin-orbit operator $H_{so}^{cp}(r, \Theta, \bar{\theta})$, under the new modified Cornell model as follows:

$$H_{pert}^{cp}(r, \Theta, \bar{\theta}) \to H_{so}^{cp}(r, \Theta, \bar{\theta}) \equiv \gamma\left\{-\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right) + \frac{\bar{\theta}}{2\mu}\right\}\vec{L}\vec{S} \tag{24}$$

here $\Theta = \sqrt{{\Theta_{12}}^2 + {\Theta_{23}}^2 + {\Theta_{13}}^2}$, $\bar{\theta} = \sqrt{{\bar{\theta}_{12}}^2 + {\bar{\theta}_{23}}^2 + {\bar{\theta}_{13}}^2}$ and γ is a new constant, which play the role of the strong coupling constant in the quantum chromodynamics or QCD theory, we have chosen the two vectors $\vec{\Theta}$ and $\vec{\bar{\theta}}$ parallel to the spin $\vec{S}$ of heavy quarkonium system. Furthermore, the above perturbative terms $H_{pert}^{cp}(r)$ can be rewritten to the following new form:

$$H_{so}^{cp}(r, \Theta, \bar{\theta}) = -\frac{\gamma}{2}\left\{\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right)\Theta - \frac{\bar{\theta}}{2\mu}\right\}(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2) \tag{25}$$

where $\vec{J}$ and $\vec{S}$ are defined as the operators of the total angular momentum and spin of quarkonium systems. This operator, $H_{so}^{cp}(r,\Theta,\bar{\theta})$, traduces the coupling between spin $\vec{S}$ and orbit momentum $\vec{L}$. The set $(H_{nc}^{cp}(r,\Theta,\bar{\theta}), J^2, L^2, S^2$ and $J_z)$ forms a complete of conserved physics quantities. For spin $-$ 1, the eigenvalues of the spin-orbit coupling operator are $k(l) \equiv \frac{1}{2}\{j(j+1) - l(l+1) - 2\}$ corresponding $j = l + 1$ (spin great), $j = l$ (spin middle) and $j = l - 1$ (spin little), respectively, then, one can form a diagonal (3×3)matrix for NMCP in 3D-NCSP symmetries, with diagonal elements$\left(H_{so}^{cp}\right)_{11}$, $\left(H_{so}^{cp}\right)_{22}$ and $\left(H_{so}^{cp}\right)_{33}$ are given by:

$$\begin{cases} \left(H_{so}^{cp}\right)_{11} = -\gamma k_1(l)\left(\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right)\Theta - \frac{\bar{\theta}}{2\mu}\right) \text{if } j = l+1 \\ \left(H_{so}^{cp}\right)_{22} = -\gamma k_2(l)\left(\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right)\Theta - \frac{\bar{\theta}}{2\mu}\right) \text{if } j = l \\ \left(H_{so}^{cp}\right)_{33} = -\gamma k_3(l)\left(\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right)\Theta - \frac{\bar{\theta}}{2\mu}\right) \text{if } j = l-1 \end{cases} \tag{26}$$

here $\left(k_1(l), k_2(l), k_3(l)\right) \equiv \frac{1}{2}(l, -2, -2l-2)$ and j is the total quantum number. The non-null diagonal elements $\left(H_{so}^{cp}\right)_{11}$, $\left(H_{so}^{cp}\right)_{22}$and $\left(H_{so}^{cp}\right)_{33}$of the modified Hamiltonian operator $H_{nc}^{cp}(\hat{r})$ will change the energy values $E_{nm}$by creating three new values:

$$\begin{cases} E_{sog}^{cp} = \langle\Psi(r,\theta,\phi)\left|\left(H_{so}^{cp}\right)_{11}\right|\Psi(r,\theta,\phi)\rangle \\ E_{som}^{cp} = \langle\Psi(r,\theta,\phi)\left|\left(H_{so}^{cp}\right)_{22}\right|\Psi(r,\theta,\phi)\rangle \\ E_{sol}^{cp} = \langle\Psi(r,\theta,\phi)\left|\left(H_{so}^{cp}\right)_{33}\right|\Psi(r,\theta,\phi)\rangle \end{cases} \tag{27}$$

We will see them in detail in the next subsection. After profound calculation, one can show that the new radial function $R_{nl}(r)$satisfying the following differential equation for modified Cornell potential:

$$\frac{d^2R_{nl}(r)}{dr^2} + 2\mu\left[E_{nl} - A - Cr + \frac{b}{r} + Dr^2 + \left(\frac{b}{2r^3} + \frac{C}{2r} - D\right)\vec{L}\vec{\Theta} - \frac{\vec{L\theta}}{2\mu} - \frac{l(l+1)}{2\mu r^2}\right]R_{nl}(r) = 0 \tag{28}$$

Through our observation of the expression of $H_{pert}^{cp}(r)$, which appear in the equation (23), we see it as proportionate to two infinitesimals parameters (Θ and$\bar{\theta}$), thus, in what follows, we proceed to solve the modified radial part of the MSE that is, equation (28) by applying standard perturbation theory to find acceptable solutions at the first order of two parameters Θ and $\overline{\theta}$. The proposed solutions for MSE under NMCP include energy corrections, which are produced automatically from two principal physical phoneme's, the first one is the effect of modified spin-orbit interaction and the second is the modified Zeeman effect while the stark effect can appear in the linear part of new modified Cornell model.

3.3 The exact modified spin-orbit spectrum for heavy quarkonium system under NMCP in pNRQCD

The purpose here is to give a complete prescription to determine the energy level of the ground state, the first excited state and the n^{th}excited state, of heavy quarkonium systems. We first find the corrections $(E_{sog}^{cp}(k_1(l), b, C, D, n, D_2, D_{1n})$, $E_{som}^{cp}(k_2(l), b, C, D, n, D_2, D_{1n})$ and $E_{sol}^{cp}(k_3(l), b, C, D, n, D_2, D_{1n}))$ for heavy quarkonium system such as (charmonium and bottomonium) mesons that have the quark and antiquark flavor under new modified Cornell potential at finite temperature, which have three polarities up and down $j = l + 1$ (spin great), $j = l$ (spin middle) and $j = l - 1$ (spin little), respectively, at the first order of two parameters (Θ and $\bar{\theta}$). Moreover, by applying the perturbative theory, in the case of perturbed nonrelativistic quantum chromodynamics pNRQCD, we obtained the following results:

$$E_{sog}^{cp} = -\gamma {C_{nl}}^2 k_1(l)$$
$$\int_0^{+\infty} r^{-\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(2\sqrt{2D_{1n}}r\right)\left\{\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n+\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(-2\sqrt{2D_{1n}}r\right)\right)\right\}^2\left(\left(\frac{b}{2r^3}+\frac{C}{2r}-D\right)\Theta-\frac{\bar{\theta}}{2\mu}\right)dr$$
$$E_{som}^{cp} = -\gamma {C_{nl}}^2 k_2(l)$$
$$\int_0^{+\infty} r^{-\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(2\sqrt{2D_{1n}}r\right)\left\{\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n+\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(-2\sqrt{2D_{1n}}r\right)\right)\right\}^2\left(\left(\frac{b}{2r^3}+\frac{C}{2r}-D\right)\Theta-\frac{\bar{\theta}}{2\mu}\right)dr$$
$$E_{sol}^{cp} = -\gamma {C_{nl}}^2 k_3(l)$$
$$\int_0^{+\infty} r^{-\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(2\sqrt{2D_{1n}}r\right)\left\{\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n+\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(-2\sqrt{2D_{1n}}r\right)\right)\right\}^2\left(\left(\frac{b}{2r^3}+\frac{C}{2r}-D\right)\Theta-\frac{\bar{\theta}}{2\mu}\right)dr \quad (29)$$

We have used the orthogonality property of the spherical harmonics $\int Y_l^m(\theta,\phi)Y_{l'}^{m'}(\theta,\phi)\,sin(\theta)\,d\theta d\phi = \delta_{ll'}\delta_{mm'}$. Now, we can rewrite the above equations to the simplified new form:

$$E_{so-gcp}(k_1, A, b, C, D, n, D_2, D_{1n}) =$$
$$-\gamma {C_{nl}}^2 k_1(l)\left\{\Theta[T_1(b,n,D_2,D_{1n})+T_2(C,n,D_2,D_{1n})+T_3(D,n,D_2,D_{1n})]-\frac{\bar{\theta}}{2\mu}T_4(n,D_2,D_{1n})\right\}$$
$$E_{so-mcp}(k_2, A, b, C, D, n, D_2, D_{1n}) =$$
$$-\gamma {C_{nl}}^2 k_2(l)\left\{\Theta[T_1(b,n,D_2,D_{1n})+T_2(C,n,D_2,D_{1n})+T_3(D,n,D_2,D_{1n})]-\frac{\bar{\theta}}{2\mu}T_4(n,D_2,D_{1n})\right\}$$
$$E_{so-lcp}(k_3, A, b, C, D, n, D_2, D_{1n}) =$$
$$-\gamma {C_{nl}}^2 k_3(l)\left\{\Theta[T_1(b,n,D_2,D_{1n})+T_2(C,n,D_2,D_{1n})+T_3(D,n,D_2,D_{1n})]-\frac{\bar{\theta}}{2\mu}T_4(n,D_2,D_{1n})\right\} \quad (30)$$

Moreover, the expressions of the 4-factors $T_i\left(i=\overline{1,4}\right)$ are given by:

$$T_1(b,n,D_2,D_{1n}) =$$
$$\frac{b}{2}\int_0^{+\infty} r^{-\frac{2D_2}{\sqrt{2D_{1n}}}-3} exp\left(2\sqrt{2D_{1n}}r\right)\left\{\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n+\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(-2\sqrt{2D_{1n}}r\right)\right)\right\}^2 dr$$
$$T_2(C,n,D_2,D_{1n}) = \frac{C}{2}$$
$$\int_0^{+\infty} r^{-\frac{2D_2}{\sqrt{2D_{1n}}}-1} exp\left(2\sqrt{2D_{1n}}r\right)\left\{\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n+\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(-2\sqrt{2D_{1n}}r\right)\right)\right\}^2 dr$$
$$T_3(D,n,D_2,D_{1n}) = -D$$
$$\int_0^{+\infty} r^{-\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(2\sqrt{2D_{1n}}r\right)\left\{\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n+\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(-2\sqrt{2D_{1n}}r\right)\right)\right\}^2 dr$$
$$T_4(n,D_2,D_{1n}) =$$
$$-\int_0^{+\infty} r^{-\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(2\sqrt{2D_{1n}}r\right)\left\{\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n+\frac{2D_2}{\sqrt{2D_{1n}}}} exp\left(-2\sqrt{2D_{1n}}r\right)\right)\right\}^2 dr \quad (31)$$

For the ground state, we have, the expressions of the 4-factors $T_i\left(i=\overline{1,4}\right)$ will be simplified to the following form:

$$T_1(b, n=0, D_2, D_{1n}) = \frac{b}{2}\int_0^{+\infty} r^{\lambda_0-2-1} exp(-\beta_0 r)\,dr$$

$$T_2(C, n=0, D_2, D_{1n}) = \frac{C}{2}\int_0^{+\infty} r^{\lambda_0-1}\, exp(-\beta_0 r)\, dr\; T_3(D, n=0, D_2, D_{1n})$$
$$= -D\int_0^{+\infty} r^{\lambda_0+1-1}\, exp(-\beta_0 r)\, dr = -DT_4(n=0, D_2, D_{1n}) \quad (32)$$

where $D_{10} = -\mu\left(E_0 - A - \frac{3C}{\delta} + \frac{6D}{\delta^2}\right)$, $\beta_0 = 2\sqrt{2D_{10}}$ and$\lambda_0 = \frac{2D_2}{\sqrt{2D_{10}}}$. It is convenient to apply the following special integral [34]:

$$\int_0^{+\infty} x^{\nu-1}\, exp(-\beta x^p)\, dx = \frac{\beta^{-\frac{\nu}{p}}}{p}\Gamma\left(\frac{\nu}{p}\right) \quad (33)$$

With conditions ($Re\,\beta\rangle 0$, $Re\,\nu\rangle 0$ and$p\rangle 0$) while $\Gamma\left(\frac{\nu}{p}\right)$ denoting to the ordinary Gamma function. After straightforward calculations, we can obtain explicit results:

$$T_1(b, n=0, D_2, D_{10}) = \frac{b}{2}{\beta_0}^{-(\lambda_0-2)}\Gamma(\lambda_0-2)$$
$$T_2(C, n=0, D_2, D_{10}) = \frac{C}{2}{\beta_0}^{-\lambda_0}\Gamma(\lambda_0)$$
$$T_3(D, n=0, D_2, D_{10}) = -D{\beta_0}^{-(\lambda_0+1)}\Gamma(\lambda_0+1) = -DT_4(n=0, D_2, D_{10}) \quad (34)$$

Allows us the two to obtain the exact modifications$E_{sog}^{cp}(k_1, A, b, C, D, n=0, D_2, D_{10})$, $E_{som}^{cp}(k_2, A, b, C, D, n=0, D_2, D_{10})$ and $E_{sol}^{cp}(k_3, A, b, C, D, n=0, D_2, D_{10})$ of the ground state as:

$$E_{sog}^{cp}(k_1, A, b, C, D, n=0, D_2, D_{10})$$
$$= -\gamma {C_{00}}^2 k_1(l=0)\left\{\Theta T_{01}(b, C, D, n=0, D_2, D_{10}) - \frac{\bar{\theta}}{2\mu}T_4(n=0, D_2, D_{10})\right\}$$
$$E_{som}^{cp}(k_2, A, b, C, D, n=0, D_2, D_{10})$$
$$= -\gamma {C_{00}}^2 k_2(l=0)\left\{\Theta T_{01}(b, C, D, n=0, D_2, D_{10}) - \frac{\bar{\theta}}{2\mu}T_5(n=0, D_2, D_{10})\right\}$$
$$E_{sol}^{cp}(k_3, A, b, C, D, n=0, D_2, D_{10}) =$$
$$-\gamma {C_{00}}^2 k_3(l=0)\left\{\Theta T_{01}(b, C, D, n=0, D_2, D_{10}) - \frac{\bar{\theta}}{2\mu}T_5(n=0, D_2, D_{10})\right\} \quad (35)$$

with

$$T_{01}(b, C, D, n=0, D_2, D_{10}) = T_1(b, n=0, D_2, D_{10}) + T_2(C, n=0, D_2, D_{10}) + T_3(D, n=0, D_2, D_{10}).$$

For the first excited state, the expressions of the 4-factors$T_i\left(i = \overline{1,4}\right)$ are given by:

$$T_1(b, n=1, D_2, D_{11}) =$$
$$\frac{b}{2}\int_0^{+\infty}\left\{{\alpha_1}^2 r^{\lambda_1-4-1}\, exp(-\beta_1 r) + {\beta_1}^2 r^{\lambda_1-2-1}\, exp(-\beta_1 r) - 2\beta_1\alpha_1 r^{\lambda_1-3-1}\, exp(-\beta_1 r)\right\}dr$$
$$T_2(C, n=1, D_2, D_{11}) =$$
$$\frac{C}{2}\int_0^{+\infty}\left\{{\alpha_1}^2 r^{\lambda_1-2-1}\, exp(-\beta_1 r) + {\beta_1}^2 r^{\lambda_1-1}1\, exp(-\beta_1 r) - 2\beta_1\alpha_1 r^{\lambda_1-1-1}\, exp(-\beta_1 r)\right\}dr$$
$$T_3(D, n=1, D_2, D_{11}) =$$
$$-D\int_0^{+\infty}\left\{{\alpha_1}^2 r^{\lambda_1-1-1}\, exp(-\beta_1 r) + {\beta_1}^2 r^{\lambda_1+1-1}\, exp(-\beta_1 r) - 2\beta_1\alpha_1 r^{\lambda_1-1}\, exp(-\beta_1 r)\right\}dr$$
$$T_4(n=1, D_2, D_{11}) =$$
$$-\int_0^{+\infty}\left\{{\alpha_1}^2 r^{\lambda_1-1-1}\, exp(-\beta_1 r) + {\beta_1}^2 r^{\lambda_1+1-1}\, exp(-\beta_1 r) - 2\beta_1\alpha_1 r^{\lambda_1-1}\, exp(-\beta_1 r)\right\}dr \quad (36)$$

where $D_{11}=-\mu\left(E_1-A-\frac{3C}{\delta}+\frac{6D}{\delta^2}\right)$, $\beta_1=2\sqrt{2D_{11}}$, $\lambda_1=\frac{2D_2}{\sqrt{2D_{11}}}$ and $\alpha_1=\lambda_1-2$. Evaluating the integral in Eq. (36) applies the special integration, which given by Eq. (33), we obtain the results:

$$T_1(b,n=1,D_2,D_{11})=\frac{b}{2}\beta_1{}^{-\lambda_1+4}\{\alpha_1{}^2\Gamma(\lambda_1-4)+\Gamma(\lambda_1-2)-2\alpha_1\Gamma(\lambda_1-3)\}$$
$$T_2(C,n=1,D_2,D_{11})=\frac{C}{2}\beta_1{}^{-\lambda_1+2}\{\alpha_1{}^2\Gamma(\lambda_1-2)+\Gamma(\lambda_1)-2\alpha_1\Gamma(\lambda_1+1)\}$$
$$T_3(D,n=1,D_2,D_{11})=-D\beta_1{}^{-\lambda_1+1}\{\alpha_1{}^2\Gamma(\lambda_1-1)+\Gamma(\lambda_1+1)-2\alpha_1\Gamma(\lambda_1)\}$$
$$=-DT_4(n=1,D_2,D_{11}) \tag{37}$$

Allows us the two to obtain the exact modifications $E_{sog}^{cp}(k_1,C,G,F,L,n=1,N,H_1)$, $E_{som}^{cp}(k_2,C,G,F,L,n=1,N,H_1)$ and $E_{sol}^{cp}(k_3,C,G,F,L,n=1,N,H_1)$ of the first excited state:

$$E_{sog}^{cp}(k_+,A,b,C,D,n=1,D_2,D_{11})$$
$$=-\gamma C_{1l}{}^2k_1(l)\left\{\Theta T_{11}(n=1,A,b,C,D,n,D_2,D_{11})-\frac{\bar{\theta}}{2\mu}T_4(n=1,N,H_1)\right\}$$
$$E_{som}^{cp}(k_-,A,b,C,D,n=1,D_2,D_{11})$$
$$=-\gamma C_{1l}{}^2k_2(l)\left\{\Theta T_{11}(n=1,A,b,C,D,n,D_2,D_{11})-\frac{\bar{\theta}}{2\mu}T_4(n=1,N,H_1)\right\}$$
$$E_{sol}^{cp}(k_-,A,b,C,D,n=1,D_2,D_{11})=$$
$$-\gamma C_{1l}{}^2k_3(l)\left\{\Theta T_{11}(n=1,A,b,C,D,n,D_2,D_{11})-\frac{\bar{\theta}}{2\mu}T_4(n=1,N,H_1)\right\} \tag{38}$$

with

$$T_{11}(b,C,D,n=1,D_2,D_{11})=T_1(b,n=1,D_2,D_{11})+T_2(C,n=1,D_2,D_{11})+T_3(D,n=1,D_2,D_{11}).$$

In the same way, we find the exact modifications $E_{sog}^{cp}(k_1,b,C,D,n,D_2,D_{1n})$, $E_{som}^{cp}(k_2,b,C,D,n,D_2,D_{1n})$ and $E_{sogl}^{cp}(k_3,b,C,D,n,D_2,D_{1n})$ for n^{th} excited states of heavy quarkonium system under new modified Cornell potential in global quantum group symmetry (3D-NCSP):

$$E_{sog}^{cp}(k_1,b,C,D,n,D_2,D_{1n})=-\gamma C_{nl}{}^2k_1(l)\left\{\Theta T_{1n}(n,b,C,D,n,D_2,D_{1n})-\frac{\bar{\theta}}{2\mu}T_4(n,D_2,D_{1n})\right\}$$
$$E_{som}^{cp}(k_2,b,C,D,n,D_2,D_{1n})=-\gamma C_{nl}{}^2k_2(l)\left\{\Theta T_{1n}(n,b,C,D,n,D_2,D_{1n})-\frac{\bar{\theta}}{2\mu}T_4(n,D_2,D_{1n})\right\}$$
$$E_{sol}^{cp}(k_3,b,C,D,n,D_2,D_{1n})=-\gamma C_{nl}{}^2k_3(l)\left\{\Theta T_{1n}(n,b,C,D,n,D_2,D_{1n})-\frac{\bar{\theta}}{2\mu}T_4(n,D_2,D_{1n})\right\} \tag{39}$$

with

$$T_{1n}(n,b,C,D,n,D_2,D_{1n})=T_1(b,n,D_2,D_{1n})+T_2(C,n,D_2,D_{1n})+T_3(D,n,D_2,D_{1n}).$$

3.4 The exact modified magnetic spectrum for heavy quarkonium systems under NMCP in pNRQCD

Further to the important previously obtained results, now, we consider another important physically meaningful phenomenon produced by the effect of new modified Cornell potential at finite temperature in perturbative NRQCD related to the influence of an external uniform magnetic field $\vec{B}$. To avoid the repetition in the theoretical calculations, it is sufficient to apply the following replacements:

$$\vec{\Theta}\to\chi\vec{B} \text{ and } \vec{\bar{\theta}}\to\overline{\sigma}\vec{B} \tag{40}$$

Allow us to make the changes $\left(-\left(\frac{b}{2r^3}+\frac{C}{2r}-D\right)\vec{\Theta}+\frac{\vec{\bar{\theta}}}{2\mu}\right)\vec{L}$ by $\left(-\left(\frac{b}{2r^3}+\frac{C}{2r}-D\right)\chi+\frac{\overline{\sigma}}{2\mu}\right)\vec{B}\vec{L}$. Here χ and $\overline{\sigma}$ are two infinitesimal real proportional constants, and we choose the arbitrary uniform external magnetic field $\vec{B}$ parallel to the (Oz) axis. This choice allows us to introduce the new modified magnetic Hamiltonian $H_m^{cp}(r,\chi,\overline{\sigma})$ in 3D-NCSP symmetries as:

$$H_{so}^{cp}(r,\Theta,\bar{\theta}) \rightarrow H_m^{cp}(r,\chi,\overline{\sigma}) = -\left(\left(\frac{b}{2r^3}+\frac{C}{2r}-D\right)\chi-\frac{\overline{\sigma}}{2\mu}\right)\aleph_{mod}^z \tag{41}$$

Here $\aleph_{mod}^z \equiv \vec{B}\vec{J}-\aleph_z$ is the new Zeeman effect and $\aleph_z \equiv -\vec{S}\vec{B}$ denote to Zeeman effect in commutative quantum mechanics. To obtain the exact NC magnetic modifications of energy for the ground state, the first excited state and n^{th} excited states of heavy quarkonium system $E_{mag}^{cp}(m=0,C,A,b,C,D,n=0,D_2,D_{10})$, $E_{mag}^{cp}(m=0,\pm1,A,b,C,D,n=1,D_2,D_{11})$ and $E_{mag}^{cp}(m=\overline{-l,+l},A,b,C,D,n,D_2,D_{1n})$ we just replace $k_1(l)$ and Θ in the Eqs. (35), (38) and (39) by the following parameters m andχ, respectively:

$$E_{mag}^{cp}(m=0,C,A,b,C,D,n=0,D_2,D_{10})=0$$
$$E_{mag}^{cp}(m=0,\pm1,A,b,C,D,n=1,D_2,D_{11})$$
$$=-\gamma C_{1l}{}^2\left\{\chi T_{11}(n=1,A,b,C,D,n=1,D_2,D_{11})-\frac{\overline{\sigma}}{2\mu}T_4(n=1,D_2,D_{11})\right\}Bm$$
$$E_{mag}^{cp}(m=\overline{-l,+l},A,b,C,D,n,D_2,D_{1n})$$
$$=-\gamma C_{nl}{}^2\left\{\chi T_{1n}(n=1,A,b,C,D,n,D_2,D_{1n})-\frac{\overline{\sigma}}{2\mu}T_4(n,D_2,D_{1n})\right\}Bm \tag{42}$$

We have $(-l\le m\le +l)$, which allows us to fix $(2l+1)$ values for discreet numbers$m$. It should be noted that the results obtained in Eq. (42) could find it by direct calculation $E_{mag}^{cp}=\langle\Psi(r,\theta,\phi)|H_m^{cp}(r,\chi,\overline{\sigma})|\Psi(r,\theta,\phi)\rangle$that takes the following explicit relation:

$$E_{mag}^{cp}=-\gamma C_{nl}{}^2mB$$
$$\int_0^{+\infty} r^{-\frac{N}{\sqrt{H_n}}+2}exp\left(2\sqrt{H_n}r\right)\left\{\left(-r^2\frac{d}{dr}\right)^n\left(r^{-2n-\frac{N}{\sqrt{H_n}}}exp\left(-2\sqrt{H_n}r\right)\right)\right\}^2\left(\left(\frac{b}{2r^3}+\frac{C}{2r}-D\right)\chi-\frac{\overline{\sigma}}{2\mu}\right)dr \tag{43}$$

Then we find the corrections produced by the operator $H_m^{cp}(r,\chi,\overline{\sigma})$ for the ground state and other excited states repeating the same calculations in the previous subsection.

4. Main Results

In the previous sub-sections, we obtained the solution of the modified Schrödinger equation for new modified Cornell potential, which is given in Eq. (20) by using the generalized Bopp's shift method and standard perturbation theory in pNRQCD. The energy eigenvalue is calculated in a three-dimensional noncommutative space phase. The modified eigenenergies $\left(E_{nc-g}^{cp},E_{nc-m}^{cp},E_{nc-l}^{cp}\right)(n=0,m=0,b,C,D,D_2,D_{10})$, $\left(E_{nc-g}^{cp},E_{nc-m}^{cp},E_{nc-l}^{cp}\right)(n=1,(m=0,\pm1),b,C,D,D_2,D_{11})$ and $\left(E_{nc-g}^{cp},E_{nc-m}^{cp},E_{nc-l}^{cp}\right)\left(n,(m=\overline{-l,+l}),b,C,D,D_2,D_{1n}\right)$ with spin-1 for MSE for heavy quarkonium systems under NMCP at finite temperature are obtained in this paper based on our original results presented on the Eqs. (35), (38), (39), and (42), in addition to the ordinary energy E_{nl} for modified Cornell potential at a finite temperature which presented in Eq. (11):

$$E_{nc-g}^{cp}(n=0,m=0,b,C,D,D_2,D_{10})=E_{00}$$
$$E_{nc-m}^{cp}(n=0,m=0,b,C,D,D_2,D_{10})$$
$$=E_{00}+\gamma C_{00}{}^2\left\{\Theta T_{10}(C,n=0,b,C,D,n=1,D_2,D_{10})-\frac{\bar{\theta}}{2\mu}T_4(n=0,D_2,D_{10})\right\}$$

$$E_{nc-l}^{cp}(n=0,m=0,b,C,D,D_2,D_{10})=E_{00}+\gamma C_{00}{}^2\left\{\Theta T_{10}(C,n=0,b,C,D,n=1,D_2,D_{10})-\frac{\bar{\theta}}{2\mu}T_4(n=0,D_2,D_{10})\right\} \tag{44}$$

and

$$E_{nc-g}^{cp}\left(n=1,k_1,(m=0,\pm1),b,C,D,D_2,D_{1_1}\right)$$
$$=E_{1l}$$
$$-\gamma {C_{1l}}^2\left\{(k_1(l=1)\Theta+\chi Bm)T_{11}(n=1,b,C,D,n=1,D_2,D_{11})\right.$$
$$\left.-\left(\frac{\bar{\theta}}{2\mu}k_1(l=1)+\frac{\overline{\sigma}}{2\mu}Bm\right)T_4(n=1,D_2,D_{11})\right\}$$
$$E_{nc-m}^{cp}\left(n=1,k_2,(m=0,\pm1),b,C,D,D_2,D_{1_1}\right)=E_{1l}-\gamma {C_{1l}}^2$$
$$\left\{(k_2(l=1)\Theta+\chi Bm)T_{11}(n=1,b,C,D,n=1,D_2,D_{11})\right.$$
$$\left.-\left(\frac{\bar{\theta}}{2\mu}k_2(l=1)+\frac{\overline{\sigma}}{2\mu}Bm\right)T_4(n=1,D_2,D_{11})\right\}$$
$$E_{nc-l}^{cp}\left(n=1,k3,(m=0,\pm1),b,C,D,D_2,D_{1_1}\right)=E_{1l}-\gamma {C_{1l}}^2\left\{(k_3(l=1)\Theta+\chi Bm)T_{11}(n=1,b,C,D,n=1,D_2,D_{11})-\left(\frac{\bar{\theta}}{2\mu}k_3(l=1)+\frac{\overline{\sigma}}{2\mu}Bm\right)T_4(n=1,D_2,D_{11})\right\} \tag{45}$$

and

$$E_{nc-g}^{cp}\left(n,k_1,m=\overline{-l,+l},b,C,D,D_2,D_{1_1}\right)=E_{nl}-\gamma {C_{nl}}^2$$
$$\left\{(k_1(l)\Theta+\chi Bm)T_{11}(n,b,C,D,n=1,D_2,D_{11})-\left(\frac{\bar{\theta}}{2\mu}k_1(l)+\frac{\overline{\sigma}}{2\mu}Bm\right)T_4(n=1,D_2,D_{11})\right\}$$

$$E_{nc-m}^{cp}\left(n,k_2,m=\overline{-l,+l},b,C,D,D_2,D_{1_1}\right)=E_{nl}-\gamma {C_{nl}}^2$$
$$\left\{(k_2(l)\Theta+\chi Bm)T_{11}(n,b,C,D,n=1,D_2,D_{11})-\left(\frac{\bar{\theta}}{2\mu}k_2(l)+\frac{\overline{\sigma}}{2\mu}Bm\right)T_4(n=1,D_2,D_{11})\right\}$$

$$E_{nc-l}^{cp}\left(n,k_3,m=\overline{-l,+l},b,C,D,D_2,D_{1_1}\right)=E_{nl}-\gamma {C_{nl}}^2\left\{(k_3(l)\Theta+\chi Bm)T_{11}(n,b,C,D,n=1,D_2,D_{11})-\left(\frac{\bar{\theta}}{2\mu}k_3(l)+\frac{\overline{\sigma}}{2\mu}Bm\right)T_4(n=1,D_2,D_{11})\right\} \tag{46}$$

where E_{00} and E_{1l} are the energy of ground state and first excited state of heavy quarkonium systems in the symmetries of quantum mechanics under modified Cornell potential at finite temperature:

$$E_{00}=Z-\frac{Y}{\left[1\pm\sqrt{W}\right]^2}\quad\text{and}\quad E_{1l}=Z-\frac{Y}{\left[3\pm\sqrt{W+4((l+1/2)^2-1/4)}\right]^2} \tag{47}$$

with $Y\equiv 2\mu\left(\frac{3C}{\delta^2}-\frac{8D}{\delta^3}+b\right)$, $Z\equiv A+\frac{3C}{\delta}-\frac{6D}{\delta^3}$ and $W\equiv 1+\frac{8\mu C}{\delta^3}-24\frac{\mu D}{\delta^4}$. This is one of the main objectives of our research and by noting that, the obtained eigenvalues of energies are real's and then the NC diagonal Hamiltonian $H_{nc}^{cp}\left(x_\mu,p_\mu\right)$ is Hermitian, furthermore, it's possible to write the three elements $\left(H_{nc}^{cp}\right)_{11}$, $\left(H_{nc}^{cp}\right)_{22}$ and $\left(H_{nc}^{cp}\right)_{33}$ as follows:

$$H_{cp}\left(x_\mu,p_\mu\right)\rightarrow H_{nc}^{cp}\left(x_\mu,p_\mu\right)\equiv\begin{pmatrix}\left(H_{nc}^{cp}\right)_{11} & 0 & 0\\ 0 & \left(H_{nc}^{cp}\right)_{22} & 0\\ 0 & 0 & \left(H_{nc}^{cp}\right)_{33}\end{pmatrix} \tag{48}$$

where

$$\left(H_{nc}^{cp}\right)_{11}=-\frac{\Delta_{nc}}{2\mu}+H_{int}^{gcp},$$
$$\left(H_{nc}^{cp}\right)_{22}=-\frac{\Delta_{nc}}{2\mu}+H_{int}^{mcp},$$
$$\left(H_{nc}^{cp}\right)_{33}=-\frac{\Delta_{nc}}{2\mu}+H_{int}^{glcp}.$$

with $\frac{\Delta_{nc}}{2\mu} = \frac{\Delta - \vec{\theta}\vec{L} - \vec{\sigma}\vec{L}}{2\mu}$ and the three modified interactions elements $\left(H_{int}^{gcp}, H_{int}^{mcp}, H_{int}^{lcp}\right)$ are given by:

$$V_{cp}(r) \rightarrow \begin{cases} H_{int}^{gcp} = a(T,r)r - \frac{b(T,r)}{r} + \gamma(k_1(l)\Theta + \chi\aleph_{mod-z})\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right) \\ H_{int}^{mcp} = a(T,r)r - \frac{b(T,r)}{r} + \gamma(k_2(l)\Theta + \chi\aleph_{mod-z})\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right) \\ H_{int}^{lcp} = a(T,r)r - \frac{b(T,r)}{r} + \gamma(k_3(l)\Theta + \chi\aleph_{mod-z})\left(\frac{b}{2r^3} + \frac{C}{2r} - D\right) \end{cases} \quad (49)$$

Thus, the ordinary kinetic term for Cornell potential $(-\frac{\Delta}{2\mu})$ and ordinary interaction $a(T,r)r - \frac{b(T,r)}{r}$ are replaced by a new modified form of the kinetic term $\frac{\Delta_{nc}}{2\mu}$ and new modified interactions modified to the new form $\left(H_{int}^{gcp}, H_{int}^{mcp}\right.$ and $\left.H_{int}^{lcp}\right)$ in 3D-NCSP symmetries. On the other hand, it is evident to consider the quantum number m takes $(2l+1)$ values and we have also two values for$(j = l \pm 1, l)$, thus every state in usually three-dimensional space of energy for heavy quarkonium system under NMCP will be $3\big(2(2l+1)\big)$sub-states. To obtain the total complete degeneracy of energy level of the modified Cornell potential in 3D-NCSP symmetries, we need to sum for all allowed values of l. Total degeneracy is thus,

$$2\sum_{i=0}^{n-1}(2l+1) = 2n^2 \rightarrow 3\left(2\sum_{i=0}^{n-1}(2l+1)\right) \equiv 6n^2 \quad (50)$$

Note that the obtained new energy eigenvalues $\left(E_{nc-g}^{cp}, E_{nc-m}^{cp}, E_{nc-l}^{cp}\right)\left(n, \left(m = \overline{-l,+l}\right), b, C, D, D_2, D_{1n}\right)$ now depend on new discrete atomic quantum numbers (n, j, l, s) and$m$ in addition to the parameters $(b, C, D)$of the modified Cornell potential. It is pertinent to note that when the atoms have spin $-$ 0, the total operator can be determined from the interval$|l - s| \leq j \leq |l + s|$, which allows us to obtain the eigenvalues of the operator $(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2)$ as $k(j, l, s) \equiv 0$ and then the nonrelativistic energy spectrum $\left(E_{nc-g}^{cp}, E_{nc-m}^{cp}, E_{nc-l}^{cp}\right)\left(n, \left(m = \overline{-l,+l}\right), b, C, D, D_2, D_{1n}\right)$ reads:

$$\begin{aligned} &\left(E_{nc-g}^{cp}, E_{nc-m}^{cp}, E_{nc-l}^{cp}\right)\left(n, \left(m = \overline{-l,+l}\right), b, C, D, D_2, D_{1n}\right) = \\ &\quad E_{nl} - \gamma {C_{nl}}^2 \left\{\chi T_{1n}(l, m, b, C, D, D_2, D_{1n}) - \frac{\overline{\sigma}}{2} T_4(n, D_2, D_{1n})\right\} Bm \end{aligned} \quad (51)$$

One of the most important applications, in the extended model of pNRQCD, is to calculate the modified mass spectra of the heavy quarkonium systems (the mass of the quarkonium bound state), such as charmonium and bottomonium mesons, that have the quark and antiquark flavor in the symmetries of NCSP under NMCP at finite temperature. To achieve this goal, we generalize the traditional formula $M = 2m + E_{nl}$ to the new form:

$$\begin{aligned} M = 2m + E_{nl} \rightarrow \mathrm{M}_{\mathrm{nc}}^{\mathrm{cp}} = 2m \\ + \frac{1}{3}\left(E_{nc-g}^{cp} + E_{nc-m}^{cp} + E_{nc-l}^{cp}\right)\left(n, \left(m = \overline{-l,+l}\right), b, C, D, D_2, D_{1n}\right) \end{aligned} \quad (52)$$

here m is the bare mass of quarkonium or twice the reduced mass of the system. Moreover, $\frac{1}{3}\left(E_{nc-g}^{cp} + E_{nc-m}^{cp} + E_{nc-l}^{cp}\right)\left(n, \left(m = \overline{-l,+l}\right), b, C, D, D_2, D_{1n}\right)$ it is the non-polarized energies, which can determine from Eqs. (46) and (51). Thus, at finite temperature $T \neq 0$, the modified mass of the quarkonium system M_{nc}^{cp} we obtain:

$$M_{nc}^{cp} = M - \gamma {C_{nl}}^2$$

$$\begin{cases} \left\{\left(\chi Bm - \frac{l+4}{6}\Theta +\right) T_{11}(n, b, C, D, D_2, D_{11}) - \left(\frac{\overline{\sigma}}{2\mu} Bm - (l+4)\frac{\bar{\theta}}{12\mu}\right) T_4(n, D_2, D_{11})\right\} & \text{For spin} - 1 \\ \left\{\chi T_{1n}(l, m, b, C, D, D_2, D_{1n}) - \frac{\overline{\sigma}}{2} T_4(n, D_2, D_{1n})\right\} Bm & \text{For Spin} - 0 \end{cases} \quad (53)$$

Here M is the heavy quarkonium system at a finite temperature under modified Cornell potential in commutative quantum mechanics, which is defined in [11]. If we consider $(\Theta,\chi)\rightarrow(0{,}0)$, we recover the results of the commutative space of the ref. [11] obtained for the modified Cornell potential, which means that our calculations are correct. Our obtained results are in good agreement with the already existing literature in NRNCSP symmetries [35]. The novelty in this work is that the generalized Bopp's shift method successfully applies to find the solution of the 3-radial modified Schrödinger Equation at finite temperature in the symmetries of NRNCSP.

5. Conclusion

In the present work, the 3-dimensional modified Schrodinger equation is analytically solved using the generalized Bopp's shift method and standard perturbation theory. The modified Cornell potential at finite temperature is extended to include the effect of the noncommutativity space phase based on Refs. [11-12]; we resume the main obtained results:

- Ordinary modified Cornell potential at finite temperature $(A(T,r)r-\frac{B(T,r)}{r})$ were replaced by new modified interactions $H_{int}^{gcp}, H_{int}^{mcp}$ and H_{int}^{lcp} for heavy quarkonium systems,
- The ordinary kinetic term $-\frac{\Delta}{2\mu}$ modified to the new form $\frac{\Delta_{nc}}{2\mu}=\frac{\Delta-\vec{\theta}\vec{L}-\vec{\sigma}\vec{L}}{2\mu}$ for heavy quarkonium system under influence of new modified Cornell model at finite temperature,
- We obtained the perturbative corrections $\left(E_{nc-g}^{cp},E_{nc-m}^{cp},E_{nc-l}^{cp}\right)(\mathrm{n}=0,\mathrm{m}=0,\mathrm{b},\mathrm{C},\mathrm{D},\mathrm{D}_2,\mathrm{D}_{10})$, $\left(\mathrm{E}_{\mathrm{nc-g}}^{\mathrm{cp}},\mathrm{E}_{\mathrm{nc-m}}^{\mathrm{cp}},\mathrm{E}_{\mathrm{nc-l}}^{\mathrm{cp}}\right)(\mathrm{n}=1,(\mathrm{m}=0,\pm1),\mathrm{b},\mathrm{C},\mathrm{D},\mathrm{D}_2,\mathrm{D}_{11})$ and $\left(E_{nc-g}^{cp},E_{nc-m}^{cp},E_{nc-l}^{cp}\right)\left(n,\left(m=\overline{-l,+l}\right),b,C,D,D_2,D_{1n}\right)$ for the ground state, the first excited state and the n^{th}excited state with (spin-1 and spin $-$ 0) for heavy quarkonium system under influence of new modified Cornell model at finite temperature is obtained.
- We have obtained at a finite temperature ($T\neq 0$) the modified mass of the quarkonium system $\mathrm{M}_{\mathrm{nc}}^{\mathrm{cp}}$ which equal the sum of corresponding value M in CQM and two perturbative terms proportional with two parameters (Θ and $\overline{\theta}$).

Through the high-value results, which we have achieved in the present work, we hope to extend our recently work physics for further investigations of elementary particles physics and other characteristics of quarkonium at a finite temperature [11] among others.

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References

[1] Kumar, R., and Chand, F., "Asymptotic Study to the N-Dimensional Radial Schrödinger Equation for the Quark-Antiquark System", *Communications in Theoretical Physics*, 59(5) (2013) 528–532. doi:10.1088/0253-6102/59/5/02

[2] Faustov, R. N., Galkin, O. V., Tatarintsev, A. V., and Vshivtsev, A. S., "Spectral problem of the radial Schrödinger equation with confining power potentials", *Theoretical and Mathematical Physics*, 113(3) (1997) 1530–1542. doi:10.1007/bf02634513

[3] Vairo, A., "The heavy quark potential in pNRQCD", *Nuclear Physics B* - Proceedings Supplements, 86(1-3) (2000) 521–524. doi:10.1016/s0920-5632(00)00615-0

[4] VEGA, A. and FLORES, J., "Heavy quarkonium properties from Cornell potential using variational method and supersymmetric quantum mechanics", *Pramana –J. Phys*., 87:73 (2016) Page 1 of 7. DOI 10.1007/s12043-016-1278-7.

[5] Mateu, V., Ortega, P. G., Entem, D. R., and Fernández, F., "Calibrating the naïve Cornell model with NRQCD", *Eur. Phys. J. C* , 79 (4) (2019) 323. https://doi.org/10.1140/epjc/s10052-019-6808-2.

[6] Kaczmarek, O., "Recent Developments in Lattice Studies for Quarkonia", *Nuclear Physics A,* 910–911 (2013) 98–105. doi: 10.1016/j.nuclphysa.2012.12.044.

[7] Fingberg, J., "Heavy quarkonia at high temperature", *Physics Letters B*, 424(3-4) (1998) 343–354. Doi: 10.1016/s0370-2693(98)00205-6.

[8] Kuchin, S. M., and Maksimenko, N. V., "Theoretical Estimations of the Spin Averaged Mass Spectra of Heavy Quarkonia and Bc Mesons", *Univ. J. Phys. Appl*., 1(3) (2013) 295-298. DOI:10.13189/ujpa.2013.010310.

[9] Ahmadov, A. I., Aydin, C., and Uzun, O., "Bound state solution of the Schrödinger equation at finite temperature", *J. Phys.: Conf. Ser*., 1194 (2019), 012001. doi:10.1088/1742-6596/1194/1/012001

[10] Abu-Shady, M., Abdel-Karim, T. A., and Khokha, E. M., "Binding Energies and Dissociation Temperatures of Heavy Quarkonia at Finite Temperature and Chemical Potential in the N-Dimensional Space", *Advances in High Energy Physics*, (2018) 1–12. doi:10.1155/2018/7356843

[11] Abu-Shady, M., "N-dimensional Schrödinger equation at finite temperature using the Nikiforov–Uvarov method", *Journal of the Egyptian Mathematical Society*, 25(1) (2017) 86–89. doi: 10.1016/j.joems.2016.06.006

[12] Al-Jamel, A. F., and Widyan, H., "Heavy Quarkonium Mass Spectra in A Coulomb Field Plus Quadratic Potential Using Nikiforov-Uvarov Method", *Applied Physics Research*, 4(3) (2012) 94-99. doi:10.5539/apr. v4n3p94

[13] Abdelmadjid Maireche, "The Klein–Gordon equation with modified Coulomb plus inverse-square potential in the noncommutative three-dimensional space", *Modern Physics Letters A,* (2019) 2050015(11 pages). doi:10.1142/s0217732320500157.

[14] Darroodi, M., Mehraban, H., and Hassanabadi, H., "The Klein–Gordon equation with the Kratzer potential in the noncommutative space", *Modern Physics Letters A*, 33, No. 35 (2018) 1850203 (6 pages). doi:10.1142/s0217732318502036.

[15] Gnatenko, K. P., "Parameters of noncommutativity in Lie-algebraic noncommutative space", *Physical Review D.* 99(2) (2019) 026009-1. doi:10.1103/physrevd.99.026009.

[16] Gnatenko, K. P., and Tkachuk, V. M., "Weak equivalence principle in noncommutative phase space and the parameters of noncommutativity", *Physics Letters A*, 381(31) (2017) 2463–2469. doi: 10.1016/j.physleta.2017.05.056.

[17] Bertolami, O., Rosa, J. G., C. M. L. De aragao, Castorina, P., and Zappala, D., "Scaling of variables and the relation between noncommutative parameters in noncommutative space phase", *Modern Physics Letters A*, 21(10) (2006) 795–802. doi:10.1142/s0217732306019840.

[18] Abdelmadjid Maireche, "A Recent Study of Excited Energy Levels of Diatomics for Modified more General Exponential Screened Coulomb Potential: Extended Quantum Mechanics", J. Nano- Electron. Phys. 9(3) (2017) 03021. DOI 10.21272/jnep.9(3).03021

[19] Djemaï, A. E. F. and Smail, H., "On Quantum Mechanics on Noncommutative Quantum Phase Space. *Commun. Theor. Phys. (Beijing, China)*. 41(6) (2004) 837–844. doi:10.1088/0253-6102/41/6/837.

[20] YUAN Yi, LI Kang, WANG Jian-Hua and CHEN Chi-Yi, "Spin-1/2 relativistic particle in a magnetic field in NC phase space", *Chinese Physics C*. 34(5) (2010) 543–547. doi:10.1088/1674-1137/34/5/005.

[21] Bertolami, O., and Leal, P., "Aspects of phase-space noncommutative space phase", *Physics Letters* B, 750 (2015) 6–11. doi: 10.1016/j.physletb.2015.08.024.

[22] Bastos, C., Bertolami, O., Dias, N.C., and Prata, J.N., "Weyl–Wigner formulation of noncommutative space phase", *Journal of Mathematical Physics*, 49(7) (2008) 072101. doi:10.1063/1.2944996.

[23] Zhang, J., "Fractional angular momentum in non-commutative spaces", *Physics Letters B*, 584(1-2) (2004) 204–209. doi: 10.1016/j.physletb.2004.01.049.

[24] Gamboa, J., Loewe, M., and Rojas, J. C., "Noncommutative space phase", *Phys. Rev. D*, 64 (2001) 067901. DOI: https://doi.org/10.1103/PhysRevD.64.067901.

[25] Chaichian, M., Sheikh-Jabbari and Tureanu, A., "Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED", *Physical Review Letters,* 86(13) (2001) 2716–2719. doi:10.1103/physrevlett.86.2716.

[26] Heisenberg, W., Letter to R. Peierls (1930), in 'Wolfgang Pauli, Scientific Correspondence', Vol. III, p.15, Ed. K. von Meyenn (1985) Springer : Verlag).

[27] Snyder, H., "Quantized Space-Time", *Physical Review*, 71(1), 1947, 38–41. doi:10.1103/physrev.71.38

[28] Abdelmadjid Maireche, "A New Nonrelativistic Investigation for the Lowest Excitations States of Interactions in One-Electron Atoms, Muonic, Hadronic and Rydberg Atoms with Modified Inverse Power Potential", *International Frontier Science Letters*., 9 (2016) 33-46. DOI: https://doi.org/10.18052/www.scipress.com/IFSL.9.33.

[29] Abdelmadjid Maireche, "New Bound States for Modified Vibrational-Rotational Structure of Super singular Plus Coulomb Potential of the Schrödinger Equation in One-Electron Atoms", *International Letters of Chemistry Physics and Astronomy*, 73 (2017) 31-45. DOI: https://doi.org/10.18052/www.scipress.com/ILCPA.73.31.

[30] Abdelmadjid Maireche "Extended of the Schrödinger Equation with New Coulomb Potentials plus Linear and Harmonic Radial Terms in the Symmetries of Noncommutative space phase", *J. Nano- Electron. Phys*., 10 No 6 (2018) 06015-1 - 06015-7. DOI: https://doi.org/10.21272/jnep.10(6).06015.

[31] Abdelmadjid Maireche, “Investigations on the Relativistic Interactions in One-Electron Atoms with Modified Yukawa Potential for Spin 1/2 Particles”, *International Frontier Science Letters*, 11 (2017) 29-44. DOI: https://doi.org/10.18052/www.scipress.com/IFSL.11.29.

[32] Gamboa, J. Loewe, M., and Rojas, J.C., “Noncommutative quantum mechanics”, *Phys. Rev. D.* 64 (2001) 067901. https://doi.org/10.1103/PhysRevD.64.067901.

[33] Gouba, L., “A comparative review of four formulations of noncommutative quantum mechanics, Int. J. Mod. Phys. A 31(19) (2016) 1630025. https://doi.org/10.1142/s0217751x16300258

[34] Gradshteyn, I. S., and Ryzhik, I. M., “Table of Integrals, Series and Products”, 7th. ed.; Elsevier, edited by Alan Jeffrey (the University of Newcastle upon Tyne, England) and Daniel Zwillinger (Rensselaer Polytechnic Institute USA) 2007.

[35] Abdelmadjid Maireche “A theoretical investigation of nonrelativistic bound state solution at finite temperature using the sum of modified Cornell plus inverse quadratic potential”. *Sri Lankan Journal of Physics*, 21(1) (2020) 11–36. DOI: http://doi.org/10.4038/sljp.v21i1.8069