ABSTRACT: The Narumi-Katayama index $NK(G)$ and first multiplicative Zagreb index $\prod_1(G)$ of a graph $G$ are defined as the product of the degrees of the vertices of $G$ and the product of square of the degrees of the vertices of $G$, respectively. The second multiplicative Zagreb index is defined as $\prod_2(G) = \prod_{uv \in E(G)} d(u)d(v)$. In this paper, we compute the extremal $NK(G)$, $\prod_1(G)$ and $\prod_2(G)$ for the graphs with given order, number of pendant vertices and cyclomatic number.

1 INTRODUCTION

In this paper we are concerned with simple connected graphs. Let $G$ be such a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices and edges of $G$ are denoted by $n$ and $m$, respectively. In a graph $G$ the number of independent cycles is called its cyclomatic number, denoted by $\gamma$. For connected graphs, the cyclomatic number is equal to $\gamma = m - n + 1$. Recall that graphs with $\gamma = 0, 1, 2$ are referred to as trees, unicyclic graphs, and bicyclic graphs, respectively. Let $v \in V(G)$ then the degree of $v$, denoted as $d(v)$, is the number of vertices of $G$ adjacent to $v$. A vertex $v$ with $d(v) = 1$ is called a pendant vertex. A graph with $n$ vertices and $n_1$ pendant vertices will be said to be an $(n, n_1)$-graph [2].

In 1984, Narumi and Katayama [6] established a definition “simple topological index”:

$$NK(G) = \prod_{v \in V(G)} d(v)$$

In recent works on this graph invariant [1, 4, 10], the name Narumi-Katayama index is being used. In [13] You and Liu deduced extremal $NK(G)$ of trees, unicyclic graphs with given diameter and vertices and the minimal $NK(G)$ of bicyclic graphs with given vertices was obtained.

The vertex-degree-based graph invariants

$$M_1(G) = \sum_{v \in V(G)} d(v)^2$$
$$M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$$

are known under the name first and second Zagreb index, respectively. They have been conceived in the 1970s and found considerable applications in chemistry [7, 11]. The Zagreb indices were subject to a large number of mathematical studies, of which we mention only a few newest [3, 5, 14]. Todeschini et al. [8, 9] have recently proposed to consider the multiplicative variants of additive graph invariants, which applied to the Zagreb indices would lead to
\[
\prod_1(G) = \prod_{v \in V(G)} d(v)^2 \\
\prod_2(G) = \prod_{uv \in E(G)} d(u)d(v) = \prod_{v \in V(G)} d(v)^d(v)
\]

The properties of these “multiplicative Zagreb indices” have not been studied so far, and the present work is an attempt to contribute towards their better understanding.

In [2], I. Gutman et al. computed the minimal first Zagreb index of graphs with fixed number of pendant vertices. In this paper we determine the extremal values of the Narumi-Katayama, first Zagreb and second Zagreb indices of connected \((n,n_1)-graphs\) with fixed cyclomatic number and show that these bounds are tight. For other notation in this paper we refer [12].

2 EXTREMAL \((n,n_1)-GRAPHS RELATIVELY TO NARUMI-KATAYAMA INDEX\)

We need the following well-known property.

**Lemma 2.1** Let \( n,r,x_1,x_2,\ldots,x_n \) be positive integers such that \( x_i \geq r \) and

\[
\sum_{i=1}^{n} x_i = a \geq rn.
\]

Then \( \prod_{i=1}^{n} x_i \) is minimum if and only if there exists an index \( i, 1 \leq i \leq n \) such that \( x_i = a-(n-1)r \)
and \( x_j = r \) for every \( j \neq i \) and it is maximum if and only if \( x_1,\ldots,x_n \) are almost equal, i.e.,
\[
\max\{x_1,\ldots,x_n\} - \min\{x_1,\ldots,x_n\} \leq 1.
\]

Since in a connected graph a non-pendant vertex has at least degree 2 and \( NK(G) = \prod_{x \in V(G),d(x) \geq 2} d(x) \), by Lemma 2.1 we have the following consequence:

**Corollary 2.2** If \( G \) is a connected graph of order \( n \) and size \( m \) with \( n_1 \) pendant vertices \((n > n_1)\) then

\[
NK(G) \geq 2^{n-n_1-1}(2m-2n+n_1+2).
\]

This bound is achieved if \( G \) has one vertex of degree \( 2m-2n+n_1+2 \) and all other non-pendant vertices are of degree 2.

Let \( G \) be a connected \((n,n_1)-\text{graph with cyclomatic number } \gamma\). If \( \gamma = 0 \) then \( G \) is a tree
and \( 2 \leq n_1 \leq n-1 \). Otherwise we suppose that \( 0 \leq n_1 \leq n-1 \). We define the auxiliary quantities \( t \), \( n_t \) and \( n_{t+1} \) as:

\[
t = \left\lfloor \frac{n+2(\gamma-1)}{n-n_1} \right\rfloor + 1, \quad n_t = (n-n_1) \left\lfloor \frac{n+2(\gamma-1)}{n-n_1} \right\rfloor - n_1 - 2(\gamma-1),
\]

\[
n_{t+1} = n - (n-n_1) \left\lfloor \frac{n+2(\gamma-1)}{n-n_1} \right\rfloor + 2(\gamma-1).
\]

Recall that \( \left\lfloor x \right\rfloor \) is the greatest integer that is not greater then \( x \), or the integer part of \( x \).
Theorem 2.3  Let \( G \) be a connected \((n, n_1)\)-graph with cyclomatic number \( \gamma \). Then

\[ 2^{n-n_1-1}(2\gamma + n_1) \leq NK(G) \leq \left( \left\lfloor \frac{n+2(\gamma-1)}{n-n_1} \right\rfloor +1 \right)^{n_1} \left( \left\lfloor \frac{n+2(\gamma-1)}{n-n_1} \right\rfloor +2 \right)^{n_1+1}. \]

(a) For trees \((\gamma = 0)\) both lower and upper bounds are reached.

(b) For \( \gamma \geq 1 \) lower bound can be attained for \( n \geq 2\gamma + 1 + n_1 \) and upper bound for \( n \geq 3\gamma + n_1 \).

Proof. Lower bound. A graph with \( n \) vertices and cyclomatic number \( \gamma \) has size \( m = n + \gamma - 1 \), so by Corollary 2.2, we have

\[ NK(G) \geq 2^{n-n_1-1}(2\gamma + n_1). \]

To see that this bound can be reached for \( \gamma = 0 \) consider a path with \( n - n_1 + 1 \) vertices and add \( n_1 - 1 \) pendant vertices, all adjacent to a unique end vertex of this path.

For \( \gamma > 0 \) take \( \gamma \) cycles, having together \( n - n_1 \) vertices and a unique common vertex. Then all \( n_1 \) remaining vertices are joined each by an edge to this common vertex. It follows that \( n \geq 2\gamma + 1 + n_1 \), and equality holds when all \( \gamma \) cycles have a length equal to 3. Figures 1, 3 and 5 illustrate graphs having minimum \( NK \) index for \( n = 22, n_1 = 13 \) and \( \gamma = 0, 1, 2 \).

Upper bound. By Lemma 2.1 \( NK(G) \) will be maximum if \( G \) has \( n_t \) \((0 < n_t \leq n - n_1)\) non-pendant vertices of degree \( t \) and \( n_{t+1} = n - n_1 - n_t \) non-pendant vertices of degree \( t + 1 \), then

\[ NK(G) \leq t^n t^{(t+1)} t_{t+1}. \]

As a graph of order \( n \) with cyclomatic number \( k \) has size \( n + \gamma - 1 \), we can write:

\[ n_1 + t n_t + (t+1)(n-n_1-n_t) = 2(n + \gamma - 1), \]

which yields

\[ t(n-n_1) - n_t = n + 2(\gamma - 1), \]

or

\[ n - \frac{n_t}{n-n_1} = \frac{n + 2(\gamma - 1)}{n-n_1}. \]

Taking integer parts,

\[ \left\lfloor \frac{n - \frac{n_t}{n-n_1}}{n-n_1} \right\rfloor = \left\lfloor \frac{n + 2(\gamma - 1)}{n-n_1} \right\rfloor. \]

Since \( t \) is a positive integer, we obtain

\[ t - 1 = \left\lfloor \frac{n + 2(\gamma - 1)}{n-n_1} \right\rfloor, \]

or

\[ t = \left\lfloor \frac{n + 2(\gamma - 1)}{n-n_1} \right\rfloor + 1. \]

From equations (3) and (4),

\[ (n-n_1)\left\lfloor \frac{n + 2(\gamma - 1)}{n-n_1} \right\rfloor + n - n_1 - n_t = n + 2(\gamma - 1), \]

which gives
As $n_{t+1} = n - n_1 - n_t$, we get:

$$n_{t+1} = n + 2(\gamma - 1) - (n - n_1) \left[\frac{n + 2(\gamma - 1)}{n - n_1}\right].$$

From equations (1) and (4) we deduce:

$$NK(G) \leq \left(\left\lceil\frac{n + 2(\gamma - 1)}{n - n_1}\right\rceil + 1\right)^{n_1} \left(\left\lceil\frac{n + 2(\gamma - 1)}{n - n_1}\right\rceil + 2\right)^{n_{t+1}},$$

as required.

For $\gamma = 0$ the upper bound can be reached. To see this consider a path $P_{n-n_1}$ with $n - n_1$ vertices. Now add the remaining $n_1$ pendant vertices using the following algorithm: join each new vertex sequentially, to a vertex of $P_{n-n_1}$, having minimum degree. Initially, all vertices have degrees 1 and 2 and after that we obtain, by construction, a tree with $n_1$ pendant vertices and non-pendant vertices having almost equal degrees.

For $\gamma > 0$ take $\gamma$ vertex disjoint cycles containing together $n - n_1$ vertices and joined by edges, such that by contracting each cycle to a vertex yields a path with $\gamma$ vertices. Then join each new vertex sequentially, to a vertex on the cycles, having minimum degree. Initially all degrees are 2 and 3 and after that we obtain, by construction, almost equal degrees for non-pendant vertices. We have $n \geq 3\gamma + n_1$, and equality holds when all vertex disjoint cycles have a length equal to 3. Figures 2, 4 and 6 illustrate graphs having maximum $NK$ index for $n = 22, n_1 = 13$ and $\gamma = 0, 1, 2$.

Figure 1: Tree with $n = 22$ and $n_1 = 13$ having minimal NK index.

Figure 2: Tree with $n = 22$ and $n_1 = 13$ having maximal NK index.
Figure 3: Unicyclic graph with $n = 22$ and $n_1 = 13$ having minimal NK index.

Figure 4: Unicyclic graph with $n = 22$ and $n_1 = 13$ having maximal NK index.

Figure 5: Bicyclic graph with $n = 22$ and $n_1 = 13$ having minimal NK index.
3 Extremal \((n,n_1)\)-graphs relatively to Multiplicative Zagreb indices

Since \(\prod_1(G) = NK(G)^2\), Theorem 2.3 implies the following corollary.

**Corollary 3.1** Let \(G\) be a connected \((n,n_1)\)-graph with cyclomatic number \(\gamma\). Then

\[
4^{n-n_1-1}(2\gamma + n_1)^2 \leq \prod_1(G) \leq \left(\frac{n+2(\gamma-1)}{n-n_1}\right)^{2n_t+1} \left(\left\lceil \frac{n+2(\gamma-1)}{n-n_1}\right\rceil + 2\right)^{2n_t+1}.
\]

(a) For trees \((\gamma = 0)\) both lower and upper bounds are reached.

(b) For \(\gamma \geq 1\) lower bound can be attained for \(n \geq 2\gamma + 1 + n_1\) and upper bound for \(n \geq 3\gamma + n_1\).

Some extremal properties of the second multiplicative Zagreb index in some families of graphs are deduced below. First we need the following property:

**Lemma 3.2** Function \(\phi(x) = \frac{x^x}{(x-1)^{x-1}}\) is increasing for \(x \geq 2\).

**Proof.** We get

\[\phi'(x) = \frac{x^x (x-1)^{x-1} (\ln x - \ln(x-1))}{(x-1)^{2(x-1)}} > 0,\]

therefore \(\phi\) is increasing for \(x \geq 2\). \(\Box\)

Let \(\Gamma_{n,n_1}\) be the family of connected graphs with order \(n\) and \(n_1\) pendant vertices.

We define a family of trees of order \(n\) with \(n_1\) pendant vertices, denoted \(T_{n,n_1}^*\) as the set of trees of order \(n\) consisting of \(n_1\) paths having a common end vertex. Note that \(T_{n,2}^* = \{P_2\}\).

**Theorem 3.3** Let \(T\) be a tree in \(\Gamma_{n,n_1}\), where \(n \geq n_1 \geq 2\); then

\[\prod_2(T) \leq n_1^{n_1} 4^{n-n_1-1}\]

and the equality holds if and only if \(T \in T_{n,n_1}^*\).
Proof. We shall prove this result by induction on \( n + n_1 \). Let \( f(n, n_1) = n_1^{n_1} 4^{n-n_1-1} \). If \( n_1 = 2 \) and \( n \geq n_1 \), then \( T \cong P_n \) and by direct calculation \( \prod_2(P_n) = 4^{n-2} \) and this equals \( f(n, 2) \), hence the property is verified.

Let \( n_1 \geq 3 \) and suppose that the result is true for any tree of order \( n' \) with \( n'_1 \) pendant vertices such that \( 7 \leq n' + n'_v < n + n_1 \). Let \( T \) be a tree in \( \Gamma_{n, n_1} \) and \( x \) be a pendant vertex of \( T \). If \( xy \in E(T) \), suppose that \( d(y) = a \). We shall consider two cases: 1) \( a = 2 \) and 2) \( a \geq 3 \).

1) In this case \( T - x \) has order \( n - 1 \) and \( n_1 \) pendant vertices, hence
\[
\prod_2(T) = 2^2 \prod_2(T - x)
\]
and by the induction hypothesis \( \prod_2(T - x) \leq f(n - 1, n_1) \) and the equality holds if and only if \( T - x \in T^{*}_{n-1, n_1} \). It follows that \( \prod_2(T) \leq f(n, n_1) \) and the equality holds if and only if \( T \in T^{*}_{n, n_1} \).

2) \( T - x \) having order \( n - 1 \) and \( n_1 - 1 \) pendant vertices, we have
\[
\prod_2(T) = \frac{a^a}{(a-1)^{a-1}} \prod_2(T - x).
\]
By our supposition of induction,
\[
\prod_2(T) \leq \frac{a^a}{(a-1)^{a-1}} f(n - 1, n_1 - 1).
\]
Equality holds if and only if \( T - x \in T^{*}_{n-1, n_1-1} \). The last inequality may be written
\[
\prod_2(T) \leq \frac{a^a}{(a-1)^{a-1}} \frac{(n_1 - 1) n_1^{n_1 - 1}}{n_1^{n_1 - 1}} f(n, n_1).
\]
By Lemma 3.1, \( \phi(x) \) is an increasing function, so \( \frac{a^a}{(a-1)^{a-1}} \) is maximum for maximum value of \( a \) and in the set of trees with \( n_1 \) pendant vertices the maximum degree of a vertex is \( n_1 \), so
\[
\prod_2(T) \leq f(n, n_1).
\]
Equality holds if and only if \( T \in T^{*}_{n, n_1} \) since \( d(y) = n_1 \) only if \( x \) is adjacent to the unique vertex in \( T - x \) of degree \( n_1 - 1 \).

We define a family of unicyclic graphs of order \( n \) with \( n_1 \) pendant vertices, denoted \( U^{*}_{n, n_1} \), as the set of unicyclic graphs of order \( n \) consisting of a cycle \( C_p \) (\( p \geq 3 \)) and \( n_1 \) paths having a common end vertex which lies on \( C_p \).

**Theorem 3.4** Let \( U \) be a unicyclic graph in \( \Gamma_{n, n_1} \) such that \( n > n_1 \geq 0 \); then
\[
\prod_2(U) \leq (n_1 + 2)^{n_1 + 2} 4^{n-n_1-1}
\]
and the equality holds if and only if \( U \in U^{*}_{n, n_1} \).

Proof. We shall prove this result also by induction on \( n + n_1 \). Let \( g(n, n_1) = (n_1 + 2)^{n_1 + 2} 4^{n-n_1-1} \). If \( n_1 = 0 \), then \( U \cong C_n \) and by direct calculation \( \prod_2(C_n) = 4^n = g(n, 0) \).
Let $n_1 \geq 1$ and suppose that the result is true for any unicyclic graph of order $n'$ with $n_1$ pendant vertices, such that $4 \leq n' + n_1 < n + n_1$. As before, let $U$ be a unicyclic graph in $\Gamma_{n,n_1}$ and $x$ a pendant vertex of $U$. If $y$ is adjacent to $x$ in $U$, let $d(y) = a$. We shall consider two cases: 1) $a = 2$ and 2) $a \geq 3$.

1) In this case $U-x$ is unicyclic, has order $n-1$ and $n_1$ pendant vertices, hence
\[
\prod_2(U) = 2^2 \prod_2(U-x)
\]
and by the induction hypothesis $\prod_2(U-x) \leq g(n-1,n_1)$ and the equality holds if and only if $U-x \in U^*_{n-1,n_1}$. It follows that $\prod_2(U) \leq g(n,n_1)$ and the equality holds if and only if $U \in U^*_{n,n_1}$.

2) $U-x$ having order $n-1$ and $n_1-1$ pendant vertices, we get
\[
\prod_2(U) = \frac{a^a}{(a-1)^{a-1}} \prod_2(U-x).
\]
By induction hypothesis,
\[
\prod_2(U) \leq \frac{a^a}{(a-1)^{a-1}} g(n-1,n_1-1),
\]
and equality holds if and only if $U \in U^*_{n-1,n_1-1}$ We deduce
\[
\prod_2(U) \leq \frac{a^a}{(a-1)^{a-1}} (n_1+1)^{n_1+1} (n_1+2)^{n_1+2} g(n,n_1).
\]
By Lemma 3.1, $\phi(a)$ is strictly increasing in $a$ and in the set of unicyclic graphs with $n_1$ pendant vertices the maximum degree of a vertex is $n_1+2$, so
\[
\prod_2(U) \leq g(n,n_1).
\]
Equality holds if and only if $U \in U^*_{n,n_1}$ since $d(y) = n_1+2$ only if $x$ is adjacent to the unique vertex in $U-x$ of degree $n_1+1$, which is common to the cycle and $n_1-1$ pendant paths.

In a similar way we can generalize this result for a given cyclomatic number $\gamma \geq 2$ for every $n \geq 3\gamma + n_1$ as follows: Let $G_{n,n_1,\gamma}$ be the set of connected graphs of order $n$, having $n_1$ pendant vertices and cyclomatic number $\gamma$, consisting of $\gamma$ cycles having a common vertex $w$ and $n_1$ paths having an end vertex in $w$. If $G$ is a connected graph of order $n$, having $n_1$ pendant vertices and cyclomatic number $\gamma$, then
\[
\prod_2(G) \leq (n_1+2\gamma)^{n_1+2\gamma} 4^{n-n_1-1}
\]
and the equality holds if and only if $G \in G^*_{n,n_1,\gamma}$.

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